Model reduction for switched DAEs

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Switched DAEs

Switched DAE

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u, \quad x(t_0^-) = \mathcal{X}_0 \subseteq \mathbb{R}^n,$$
$$y = C_\sigma x + D_\sigma u, \tag{swDAE}$$

- Switching signal: $\sigma : [t_0, t_f) \to \mathcal{Q} := \{0, 1, \dots, \mathtt{m}\}$
- Modes: $(E_k, A_k, B_k, C_k, D_k)$ for $k \in \mathcal{Q}$
- Singular system: $E_k \in \mathbb{R}^{n\times n}$ usually singular

Motivation

- Electrical circuits with switches
- (Linearized) models of water distribution networks with valves
- Mathematical curiosity

Toy Example

Consider (swDAE) given by:

on $[t_0, s_1)$:

$$\dot{x} = \begin{bmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} x + \begin{bmatrix}0\\0\\1\\0\end{bmatrix} u$$

$$y = 0$$

on $[s_1, s_2)$:

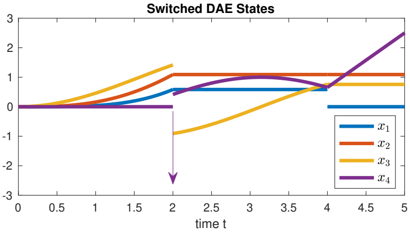
$$\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&1&0 \end{bmatrix} \dot{x} = x + \begin{bmatrix}0\\0\\1\\0\end{bmatrix} u$$

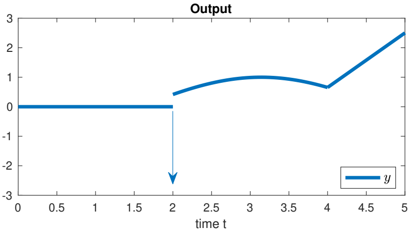
$$y = \begin{bmatrix}0&0&0&1\end{bmatrix} x$$

on $[s_2, t_f)$:

$$\begin{bmatrix} 0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix} \dot{x} = \begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&1&0 \end{bmatrix} x$$

$$y = \begin{bmatrix}0&0&0&1\end{bmatrix} x$$



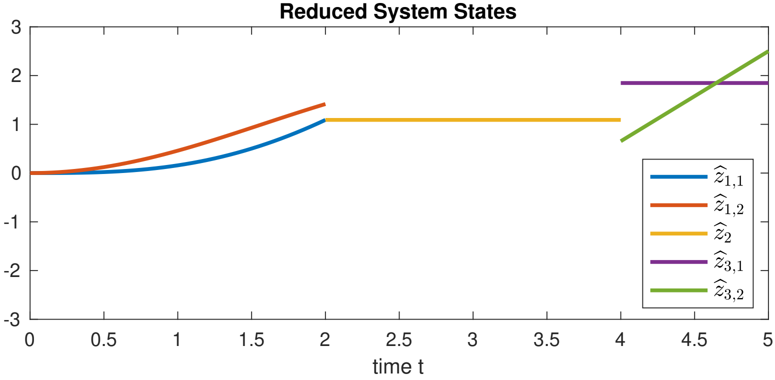


Model reduction

Model reduction task

(Approximately) same input-output behavior with smaller size switched system

For the toy example: possible to reduce to mode-dependent state-dimensions $(2, 1, 2)$:



Key challenges and novelties

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u, \quad x(t_0^-) = \mathcal{X}_0 \subseteq \mathbb{R}^n, \\ y &= C_\sigma x + D_\sigma u, \end{aligned} \tag{swDAE}$$

- Fixed switching signal on fixed finite time interval $[t_0, t_f)$
- No stability assumption for individual modes
- No restriction on index of DAE $\rightsquigarrow$ Dirac impulses in state and output
- Allow non-zero (possibly inconsistent) initial values via subspace $\mathcal{X}_0$
- Reduced model should again be a switched system (with same switching signal)
- Allow mode-dependent reduced state dimension

Overview: reduction approach

$$
\begin{aligned}
E_\sigma \dot{x} &= A_\sigma x + B_\sigma u\\
y &= C_\sigma x + D_\sigma u
\end{aligned}
$$

QWF

$$
\begin{aligned}
\dot{z} &= A_k^{\text{diff}} z + B_k^{\text{diff}} u, \quad \text{on } (s_k, s_{k+1})\\
z(s_k^+) &= \Pi_k \left[z(s_k^-) + \boldsymbol{B}_{k-1}^{\text{imp}} \boldsymbol{U}^{\nu_{k-1}}(s_k^-) \right]\\
y &= C_k z + D_k u + \boldsymbol{D}_k^{\text{imp}} \boldsymbol{U}^{\nu_k}, \quad \text{on } (s_k, s_{k+1})\\
y[s_k] &= \sum_{i=0}^{\nu_k-2} \left[C_k^i z(s_k^-) + \boldsymbol{D}_{k,i}^{\text{imp}-} \boldsymbol{U}^{\nu_{k-1}}(s_k^-) - \boldsymbol{D}_{k,i}^{\text{imp}+} \boldsymbol{U}^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)}
\end{aligned}
$$

reduced realization

$$
\begin{aligned}
\dot{\widehat{z}} &= \widehat{A}_k^{\text{diff}} \widehat{z} + \widehat{B}_k^{\text{diff}} u, \quad \text{on } (s_k, s_{k+1}),\\
\widehat{z}(s_k^+) &= \widehat{\Pi}_k \widehat{z}(s_k^-) + \widehat{J}_k^v \boldsymbol{U}^{\nu_{k-1}}(s_k^-),\\
y &= \widehat{C}_k \widehat{z} + D_k u + \boldsymbol{D}_k^{\text{imp}} \boldsymbol{U}^{\nu_k}, \quad \text{on } (s_k, s_{k+1}),\\
y[s_k] &= \sum_{i=0}^{\nu_k-2} \left[\widehat{C}_k^i \widehat{z}(s_k^-) + \boldsymbol{D}_{k,i}^{\text{imp}-} \boldsymbol{U}^{\nu_{k-1}}(s_k^-) - \boldsymbol{D}_{k,i}^{\text{imp}+} \boldsymbol{U}^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)},
\end{aligned}
$$

impulse decoupling

$$
\begin{aligned}
\dot{x}^{\overline{\text{imp}}} &= A_k^{\overline{\text{imp}}} x^{\overline{\text{imp}}} + B_k^{\overline{\text{imp}}} u, \quad \text{on } (s_k, s_{k+1}),\\
\dot{x}^{\text{imp}} &= A_k^{\text{imp}} x^{\text{imp}} + B_k^{\text{imp}} u, \quad \text{on } (s_k, s_{k+1}),\\
x^{\overline{\text{imp}}}(s_k^+) &= J_k^{x^{\overline{\text{imp}}}} x^{\overline{\text{imp}}}(s_k^-) + J_k^{v^{\overline{\text{imp}}}} \boldsymbol{U}^{\nu_{k-1}}(s_k^-),\\
x^{\text{imp}}(s_k^+) &= J_k^{x^{\text{imp}}} x^{\text{imp}}(s_k^-) + J_k^{v^{\text{imp}}} \boldsymbol{U}^{\nu_{k-1}}(s_k^-),\\
y &= C_k^{\overline{\text{imp}}} x^{\overline{\text{imp}}} + C_k^{\text{imp}} x^{\text{imp}} + D_k u + D_k^{\text{imp}} U^{\nu_k}\\
y[s_k] &= \sum_{i=0}^{\rho_k} \left[C_k^{\text{imp},i} x^{\text{imp}}(s_k^-) + D_{k,i}^{\text{imp}-} U^{\nu_{k-1}}(s_k^-) - D_{k,i}^{\text{imp}+} U^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)},
\end{aligned}
$$

midpoint balanced truncation

$$
\begin{aligned}
\dot{\widehat{x}}^{\overline{\text{imp}}} &= \widehat{A}_k^{\overline{\text{imp}}} \widehat{x}^{\overline{\text{imp}}} + \widehat{B}_k^{\overline{\text{imp}}} u, \quad \text{on } (s_k, s_{k+1}),\\
\dot{x}^{\text{imp}} &= A_k^{\text{imp}} x^{\text{imp}} + B_k^{\text{imp}} u, \quad \text{on } (s_k, s_{k+1}),\\
\widehat{x}^{\overline{\text{imp}}}(s_k^+) &= \widehat{J}_k^{\widehat{x}^{\overline{\text{imp}}}} \widehat{x}^{\overline{\text{imp}}}(s_k^-) + \widehat{J}_k^{v^{\overline{\text{imp}}}} \boldsymbol{U}^{\nu_{k-1}}(s_k^-),\\
x^{\text{imp}}(s_k^+) &= J_k^{x^{\text{imp}}} x^{\text{imp}}(s_k^-) + J_k^{v^{\text{imp}}} U^{\nu_{k-1}}(s_k^-),\\
\widehat{y} &= \widehat{C}_k^{\overline{\text{imp}}} \widehat{x}^{\overline{\text{imp}}} + C_k^{\text{imp}} x^{\text{imp}} + D_k u + D_k^{\text{imp}} U^{\nu_k}\\
\widehat{y}[s_k] &= \sum_{i=0}^{\rho_k} \left[C_k^{\text{imp},i} x^{\text{imp}}(s_k^-) + D_{k,i}^{\text{imp}-} U^{\nu_{k-1}}(s_k^-) - D_{k,i}^{\text{imp}+} U^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)}
\end{aligned}
$$

Overall reduction

The three main steps

1. **Reduced realization** (always possible, depends only on mode sequence)
   - Via Wong-sequences and Quasi-Weierstrass form rewrite (swDAE) as switched ODE with jumps and impulsive output of same size
   - Calculate extended reachability and restricted unobservability subspaces
   - Calculate weak Kalman decomposition and remove unreachable/unobservable parts
   - Define reduced jump maps, output impulses, initial value space and initial projector
2. **Impulse decoupling** (structural assumption, depends only on mode sequence)
   - Key observation: Dirac impulse = infinite peak
     $\rightsquigarrow$ do not change states which effect output Diracs
   - Assumption: States evolve in two disjoint invariant (mode-dependent) subspaces
3. **Midpoint balanced truncation** (invertability assumption on Gramians)
   - Solution = Solution for continuous input + Solution for discrete input
   - Calculate midpoint reachability Gramians for continuous and discrete time system
   - Calculate midpoint observability Gramians
   - Apply mode-wise balanced truncation via the midpoint Gramians

From (swDAE) to switched ODE

$$E_\sigma \dot x = A_\sigma x + B_\sigma u$$
$$y = C_\sigma x + D_\sigma u$$

QWF

$$\dot z = A_k^{\text{diff}} z + B_k^{\text{diff}} u, \quad \text{on } (s_k, s_{k+1})$$
$$z(s_k^+) = \Pi_k \left[z(s_k^-) + \boldsymbol{B}_{k-1}^{\text{imp}} \boldsymbol{U}^{\nu_k - 1}(s_k^-) \right]$$
$$y = C_k z + D_k u + \boldsymbol{D}_k^{\text{imp}} \boldsymbol{U}^{\nu_k}, \quad \text{on } (s_k, s_{k+1})$$
$$y[s_k] = \sum_{i=0}^{\nu_k - 2} \left[C_k^i z(s_k^-) + \boldsymbol{D}_{k,i}^{\text{imp}-} \boldsymbol{U}^{\nu_k - 1}(s_k^-) - \boldsymbol{D}_{k,i}^{\text{imp}+} \boldsymbol{U}^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)}$$

reduced realization

$$\dot{\hat z} = \hat A_k^{\text{diff}} z + \hat B_k^{\text{diff}} u, \quad \text{on } (s_k, s_{k+1}),$$
$$\hat z(s_k^+) = \hat \Pi_k \hat z(s_k^-) + \hat J_k^v \boldsymbol{U}^{\nu_k - 1}(s_k^-),$$
$$y = \hat C_k \hat z + D_k u + D_k^{\text{imp}} U^{\nu_k}, \quad \text{on } (s_k, s_{k+1}),$$
$$y[s_k] = \sum_{i=0}^{\nu_k - 2} \left[\hat C_k^i \hat z(s_k^-) + D_{k,i}^{\text{imp}-} U^{\nu_k - 1}(s_k^-) - D_{k,i}^{\text{imp}+} U^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)},$$

impulse decoupling

$$\dot x^{\overline{\text{imp}}} = A_k^{\overline{\text{imp}}} x^{\overline{\text{imp}}} + B_k^{\overline{\text{imp}}} u, \quad \text{on } (s_k, s_{k+1}),$$
$$\dot x^{\text{imp}} = A_k^{\text{imp}} x^{\text{imp}} + B_k^{\text{imp}} u, \quad \text{on } (s_k, s_{k+1}),$$
$$x^{\overline{\text{imp}}}(s_k^+) = J_k^{x^{\overline{\text{imp}}}} x^{\overline{\text{imp}}}(s_k^-) + J_k^{v^{\overline{\text{imp}}}} \boldsymbol{U}^{\nu_k - 1}(s_k^-),$$
$$x^{\text{imp}}(s_k^+) = J_k^{x^{\text{imp}}} x^{\text{imp}}(s_k^-) + J_k^{v^{\text{imp}}} \boldsymbol{U}^{\nu_k - 1}(s_k^-),$$
$$y = C_k^{\overline{\text{imp}}} x^{\overline{\text{imp}}} + C_k^{\text{imp}} x^{\text{imp}} + D_k u + D_k^{\text{imp}} U^{\nu_k}$$
$$y[s_k] = \sum_{i=0}^{\rho_k} \left[C_k^{\text{imp},i} x^{\text{imp}}(s_k^-) + D_{k,i}^{\text{imp}-} U^{\nu_k - 1}(s_k^-) - D_{k,i}^{\text{imp}+} U^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)},$$

midpoint balanced truncation

$$\dot{\hat x}^{\overline{\text{imp}}} = \hat A_k^{\overline{\text{imp}}} \hat x^{\overline{\text{imp}}} + \hat B_k^{\overline{\text{imp}}} u, \quad \text{on } (s_k, s_{k+1}),$$
$$\dot x^{\text{imp}} = A_k^{\text{imp}} x^{\text{imp}} + B_k^{\text{imp}} u, \quad \text{on } (s_k, s_{k+1}),$$
$$\hat x^{\overline{\text{imp}}}(s_k^+) = \hat J_k^{\hat x^{\overline{\text{imp}}}} \hat x^{\overline{\text{imp}}}(s_k^-) + \hat J_k^{v^{\overline{\text{imp}}}} \boldsymbol{U}^{\nu_k - 1}(s_k^-),$$
$$x^{\text{imp}}(s_k^+) = J_k^{x^{\text{imp}}} x^{\text{imp}}(s_k^-) + J_k^{v^{\text{imp}}} U^{\nu_k - 1}(s_k^-),$$
$$\hat y = \hat C_k^{\overline{\text{imp}}} \hat x^{\overline{\text{imp}}} + C_k^{\text{imp}} x^{\text{imp}} + D_k u + D_k^{\text{imp}} U^{\nu_k}$$
$$\hat y[s_k] = \sum_{i=0}^{\rho_k} \left[C_k^{\text{imp},i} x^{\text{imp}}(s_k^-) + D_{k,i}^{\text{imp}-} U^{\nu_k - 1}(s_k^-) - D_{k,i}^{\text{imp}+} U^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)}$$

Overall reduction

Some DAE fundamentals

$$E\dot{x} = Ax + Bu \qquad \text{(DAE)}$$

Definition (Regularity)

(E, A) or (DAE) is called regular $:\Longleftrightarrow \quad \det(sE - A) \not\equiv 0$

Theorem (Regularity characterizations)

(DAE) *is regular*

$\Longleftrightarrow$ $\forall u\ \exists$ *solution of* (DAE), *uniquely determined by* $x(t_0)$

$\Longleftrightarrow$ $\forall u\ \forall x_0 \in \mathbb{R}^n$ *exists unique distributional solution with* $x(t_0^-) = x_0$

$\Longleftrightarrow$ $\exists S, T$ *such that* (SET, SAT) *is in quasi-Weierstrass form*

$$\left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \quad N \text{ nilpotent} \qquad \text{(QWF)}$$

S, T *and* (QWF) *can be easily obtained via Wong-limits* $\mathcal{V}^*, \mathcal{W}^* \subseteq \mathbb{R}^n$

Wong-decomposition

Definition (Some matrix definition based on Wong limits)

$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$ $\quad \Pi^{\text{diff}}_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S$ $\quad \Pi^{\text{imp}}_{(E,A)} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S$

$A^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} A$ $\quad B^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} B$ $\quad E^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} E$ $\quad B^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} B$

Theorem (Solution decomposition)

x *solves* (DAE) *with* $x(t_0^-) = x_0 \iff x = x^{\text{diff}} + x^{\text{imp}} \in \mathcal{V}^* \oplus \mathcal{W}^*$ *where*

$$\begin{aligned} \dot{x}^{\text{diff}} &= A^{\text{diff}} x^{\text{diff}} + B^{\text{diff}} u, & x^{\text{diff}}(t_0^-) &= \Pi_{(E,A)} x_0, \\ E^{\text{imp}} \dot{x}^{\text{imp}} &= x^{\text{imp}} + B^{\text{imp}} u, & x^{\text{imp}}(t_0^-) &= (I - \Pi_{(E,A)}) x_0. \end{aligned}$$

Explicit impulsive solution formula

Lemma

x^{imp} *solves* $E^{\text{imp}}\dot{x}^{\text{imp}} = x^{\text{imp}} + B^{\text{imp}}u$, $x^{\text{imp}}(t_0^-) = (I - \Pi)x_0 \iff$

$$x^{\text{imp}} = \boldsymbol{B}^{\text{imp}}\boldsymbol{U}^{\nu} \quad \text{on } (t_0, t_f)$$

$$x^{\text{imp}}[t_0] = -\sum_{i=0}^{\nu-2}(E^{\text{imp}})^{i+1}(x_0 - \boldsymbol{B}^{\text{imp}}\boldsymbol{U}^{\nu}(t_0^+))\delta_{t_0}^{(i)},$$

where $\nu \in \mathbb{N}$ *is the nilpotency index of* E^{imp} *and*

$$\boldsymbol{U}^{\nu} := \left[u^\top, \dot{u}^\top, \cdots, u^{(\nu-1)^\top}\right]^\top$$

$$\boldsymbol{B}^{\text{imp}} := -\left[B^{\text{imp}}, E^{\text{imp}}B^{\text{imp}}, \dots, (E^{\text{imp}})^{\nu-1}B^{\text{imp}}\right].$$

Equivalent switched ODE formulation

Corollary

For each $x_0 \in \mathbb{R}^n$ *the input-output behavior of* (swDAE) *is equal to the one of*

$$\begin{aligned} \dot z &= A_k^{\text{diff}} z + B_k^{\text{diff}} u, \quad \text{on } (s_k, s_{k+1}), \quad z(t_0^-) = x_0 \\ z(s_k^+) &= \Pi_k \left[z(s_k^-) + \boldsymbol{B}_{k-1}^{\text{imp}} \boldsymbol{U}^{\nu_{k-1}}(s_k^-) \right], \quad k \geq 0 \\ y &= C_k z + D_k u + \boldsymbol{D}_k^{\text{imp}} \boldsymbol{U}^{\nu_k}, \quad \text{on } (s_k, s_{k+1}) \end{aligned}$$

$$y[s_k] = \sum_{i=0}^{\nu_k - 2} \left[C_k^i z(s_k^-) + \boldsymbol{D}_{k-1,i}^{\text{imp}-} \boldsymbol{U}^{\nu_{k-1},i}(s_k^-) - \boldsymbol{D}_k^{\text{imp}+} \boldsymbol{U}^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)}$$

where $\boldsymbol{B}_{-1}^{\text{imp}} := 0$, $\boldsymbol{D}_k^{\text{imp}} := C_k \boldsymbol{B}_k^{\text{imp}}$, $C_k^i := -C_k (E_k^{\text{imp}})^{i+1}$, $\boldsymbol{D}_{k,i}^{\text{imp}-} := -C_k (E_k^{\text{imp}})^{i+1} \boldsymbol{B}_{k-1}^{\text{imp}}$ *and* $\boldsymbol{D}_{k,i}^{\text{imp}+} := -C_k (E_k^{\text{imp}})^{i+1} \boldsymbol{B}_k^{\text{imp}}$.

Toy example - Wong matrices

The matrices $(\Pi_k, A_k^{\mathsf{diff}}, B_k^{\mathsf{diff}}, E_k^{\mathsf{imp}}, B_k^{\mathsf{imp}})$ are given by

$$
\begin{aligned}
k=0: &\quad \left(\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}\right),\\
k=1: &\quad \left(\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right),\\
k=2: &\quad \left(\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&1&0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}\right).
\end{aligned}
$$

The corresponding feedthrough terms are then

$$
\boldsymbol{D}_0^{\mathsf{imp}} = 0_{1\times 0}, \quad \boldsymbol{D}_1^{\mathsf{imp}} = [\,0\ \text{-}1\,], \quad \boldsymbol{D}_2^{\mathsf{imp}} = 0_{1\times 1}, \quad \boldsymbol{D}_{1,0}^{\mathsf{imp}+} = [\,1\ 0\,], \quad \boldsymbol{D}_{1,0}^{\mathsf{imp}-} = 0_{1\times 0}.
$$

Toy example - switched ODE representation

on (s_0, s_1) :

$$\dot z = \begin{bmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix} z + \begin{bmatrix} 0\\0\\1\\0 \end{bmatrix} u$$

$$z(s_0^+) = x_0$$

$$y = 0$$

$$y[s_0] = 0$$

on (s_1, s_2) :

$$\dot z = 0$$

$$z(s_1^+) = \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix} z(s_1^-)$$

$$y = \begin{bmatrix} 0&0&0&1 \end{bmatrix} z + \begin{bmatrix} 0&-1 \end{bmatrix} \begin{pmatrix} u\\ \dot u \end{pmatrix}$$

$$y[s_1] = \left(\begin{bmatrix} 0&0&-1&0 \end{bmatrix} z(s_1^-) - \begin{bmatrix} 1&0 \end{bmatrix} \begin{pmatrix} u(s_1^+)\\ \dot u(s_1^+) \end{pmatrix} \right) \delta_{s_1}$$

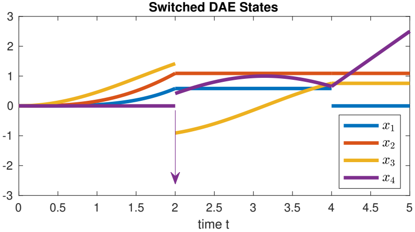
on (s_2, s_3) :

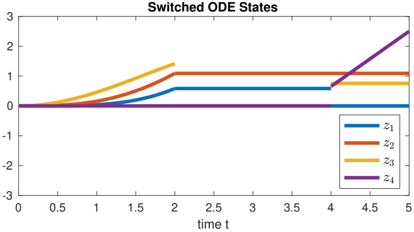
$$\dot z = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&1&1&0 \end{bmatrix} z$$

$$z(s_2^+) = \begin{bmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix} z(s_2^-) - \begin{bmatrix} 0&0\\ 0&0\\ 1&0\\ 0&1 \end{bmatrix} \begin{pmatrix} u(s_1^-)\\ \dot u(s_1^-) \end{pmatrix}$$

$$y = \begin{bmatrix} 0&0&0&1 \end{bmatrix} z$$

$$y[s_2] = 0$$





Reduced realization of switched ODE

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u$$
$$y = C_\sigma x + D_\sigma u$$

QWF

$$\dot{z} = A_k^{\text{diff}} z + B_k^{\text{diff}} u, \quad \text{on } (s_k, s_{k+1})$$
$$z(s_k^+) = \Pi_k \left[z(s_k^-) + \boldsymbol{B}_{k-1}^{\text{imp}} \boldsymbol{U}^{\nu_{k-1}}(s_k^-) \right]$$
$$y = C_k z + D_k u + \boldsymbol{D}_k^{\text{imp}} \boldsymbol{U}^{\nu_k}, \quad \text{on } (s_k, s_{k+1})$$
$$y[s_k] = \sum_{i=0}^{\nu_k - 2} \left[C_k^i z(s_k^-) + \boldsymbol{D}_{k,i}^{\text{imp}-} \boldsymbol{U}^{\nu_{k-1}}(s_k^-) - \boldsymbol{D}_{k,i}^{\text{imp}+} \boldsymbol{U}^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)}$$

reduced realization

$$\dot{\widehat{z}} = \widehat{A}_k^{\text{diff}} z + \widehat{B}_k^{\text{diff}} u, \quad \text{on } (s_k, s_{k+1}),$$
$$\widehat{z}(s_k^+) = \widehat{\Pi}_k \widehat{z}(s_k^-) + \widehat{J}_k^v \boldsymbol{U}^{\nu_{k-1}}(s_k^-),$$
$$y = \widehat{C}_k \widehat{z} + D_k u + D_k^{\text{imp}} U^{\nu_k}, \quad \text{on } (s_k, s_{k+1}),$$
$$y[s_k] = \sum_{i=0}^{\nu_k - 2} \left[\widehat{C}_k^i \widehat{z}(s_k^-) + D_{k,i}^{\text{imp}-} U^{\nu_{k-1}}(s_k^-) - D_{k,i}^{\text{imp}+} U^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)},$$

impulse decoupling

$$\dot{x}^{\overline{\text{imp}}} = A_k^{\overline{\text{imp}}} x^{\overline{\text{imp}}} + B_k^{\overline{\text{imp}}} u, \quad \text{on } (s_k, s_{k+1}),$$
$$\dot{x}^{\text{imp}} = A_k^{\text{imp}} x^{\text{imp}} + B_k^{\text{imp}} u, \quad \text{on } (s_k, s_{k+1}),$$
$$x^{\overline{\text{imp}}}(s_k^+) = J_k^{x^{\overline{\text{imp}}}} x^{\overline{\text{imp}}}(s_k^-) + J_k^{v^{\overline{\text{imp}}}} \boldsymbol{U}^{\nu_{k-1}}(s_k^-),$$
$$x^{\text{imp}}(s_k^+) = J_k^{x^{\text{imp}}} x^{\text{imp}}(s_k^-) + J_k^{v^{\text{imp}}} \boldsymbol{U}^{\nu_{k-1}}(s_k^-),$$
$$y = C_k^{\overline{\text{imp}}} x^{\overline{\text{imp}}} + C_k^{\text{imp}} x^{\text{imp}} + D_k u + D_k^{\text{imp}} U^{\nu_k}$$
$$y[s_k] = \sum_{i=0}^{\rho_k} \left[C_k^{\text{imp},i} x^{\text{imp}}(s_k^-) + D_{k,i}^{\text{imp}-} U^{\nu_{k-1}}(s_k^-) - D_{k,i}^{\text{imp}+} U^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)},$$

midpoint balanced truncation

$$\dot{\widehat{x}}^{\overline{\text{imp}}} = \widehat{A}_k^{\overline{\text{imp}}} \widehat{x}^{\overline{\text{imp}}} + \widehat{B}_k^{\overline{\text{imp}}} u, \quad \text{on } (s_k, s_{k+1}),$$
$$\dot{x}^{\text{imp}} = A_k^{\text{imp}} x^{\text{imp}} + B_k^{\text{imp}} u, \quad \text{on } (s_k, s_{k+1}),$$
$$\widehat{x}^{\overline{\text{imp}}}(s_k^+) = \widehat{J}_k^{x^{\overline{\text{imp}}}} \widehat{x}^{\overline{\text{imp}}}(s_k^-) + \widehat{J}_k^{v^{\overline{\text{imp}}}} \boldsymbol{U}^{\nu_{k-1}}(s_k^-),$$
$$x^{\text{imp}}(s_k^+) = J_k^{x^{\text{imp}}} x^{\text{imp}}(s_k^-) + J_k^{v^{\text{imp}}} U^{\nu_{k-1}}(s_k^-),$$
$$\widehat{y} = \widehat{C}_k^{\overline{\text{imp}}} \widehat{x}^{\overline{\text{imp}}} + C_k^{\text{imp}} x^{\text{imp}} + D_k u + D_k^{\text{imp}} U^{\nu_k}$$
$$\widehat{y}[s_k] = \sum_{i=0}^{\rho_k} \left[C_k^{\text{imp},i} x^{\text{imp}}(s_k^-) + D_{k,i}^{\text{imp}-} U^{\nu_{k-1}}(s_k^-) - D_{k,i}^{\text{imp}+} U^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)}$$

Overall reduction

Reduced realization - notation reset

$$\begin{aligned}
\dot z &= A_k z + B_k u, && \text{on } (s_k, s_{k+1}), && z(t_0^-) = x_0 \in \mathcal{X}_0,\\
z(s_k^+) &= J_k^z z(s_k^-) + J_k^v v_k, && k \geq 0,\\
y &= C_k z, && \text{on } (s_k, s_{k+1}),\\
y[s_k] &= \sum_{i=0}^{\rho_k} C_k^i z(s_k^-)\delta_{s_k}^{(i)}, && k \geq 0,
\end{aligned}$$

$\downarrow$ reduction

$$\begin{aligned}
\dot{\widehat z} &= \widehat A_k \widehat z + \widehat B_k u, && \text{on } (s_k, s_{k+1}), && \widehat z(t_0^-) = \widehat z_0(x_0),\\
\widehat z(s_k^+) &= \widehat J_k^z \widehat z(s_k^-) + \widehat J_k^v v_k, && k \geq 0,\\
y &= \widehat C_k \widehat z, && \text{on } (s_k, s_{k+1}),\\
y[s_k] &= \sum_{i=0}^{\rho_k} \widehat C_k^i \widehat z(s_k^-)\delta_{s_k}^{(i)}, && k \geq 0,
\end{aligned}$$

Recall: Kalman decomposition

Reachable subspace for $\dot{x} = Ax + Bu$

$\mathcal{R} := \langle A \mid \operatorname{im} B \rangle := \operatorname{im}[B, AB, ..., A^{n-1}B]$ ⇝ smallest A-inv. subspace containing $\operatorname{im} B$

Unobservable subspace for $\dot{x} = Ax$, $y = Cx$

$\mathcal{U} := \langle \ker C \mid A \rangle := \ker[C/CA/\dots/CA^{n-1}]$ ⇝ largest A-inv. subspace contained in $\ker C$

Kalman decomposition

Choose coordinate transformation $Q = [P^1, P^2, P^3, P^4]$ such that

$$\operatorname{im} P^1 = \mathcal{R} \cap \mathcal{U}, \quad \operatorname{im}[P^1, P^2] = \mathcal{R}, \quad \operatorname{im}[P^1, P^3] = \mathcal{U}$$

then $(Q^{-1}AQ, Q^{-1}B, CQ)$ is a Kalman decomposition:

$$\left(\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & C_2 & 0 & C_4 \end{bmatrix} \right)$$

⇝ (A_{22}, B_2, C_2) has same input-output behavior as (A, B, C) for $x_0 \in \mathcal{R}$

Removing unreachable/unobservable states

Reduced realization: Basic idea

Remove unreachable/unobservable states
$\rightsquigarrow$ reduced system with same input-output behavior

Challenges for switched DAE

- Structurally unreachable: States evolve within consistency subspace
- Initial value before switch structurally unreachable for current mode
- Reachable and unobservable subspaces fully time-varying for switched systems

Example to illustrate time-varying nature of reachable space:

$$\dot{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \text{ on } [t_0, s_1), \qquad \dot{x} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \text{ on } [s_1, t_f)$$

$$\mathcal{R}_{[t_0,t)} = \operatorname{im} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ for } t \in (t_0, t_1], \qquad \mathcal{R}_{[t_0,t)} = \operatorname{im} \begin{bmatrix} \cos(t-s_1) & 0 \\ \sin(t-s_1) & 0 \\ 0 & 1 \end{bmatrix} \text{ for } t \in (s_1, t_f)$$

Weak Kalman decomposition

Definition

- $\overline{\mathcal{R}} \subseteq \mathbb{R}^n$ is called extended reachable subspace
  $:\Longleftrightarrow \overline{\mathcal{R}}$ is A-invariant and contains $\operatorname{im} B$ (and hence $\mathcal{R}$)
- $\underline{\mathcal{U}} \subseteq \mathbb{R}^n$ is called restricted unobservable subspace
  $:\Longleftrightarrow \underline{\mathcal{U}}$ is A-invariant and is contained in $\ker C$ (and hence in $\mathcal{U}$)

Weak Kalman decomposition

Choose coordinate transformation $Q = [P^1, P^2, P^3, P^4]$ such that

$$\operatorname{im} P^1 = \overline{\mathcal{R}} \cap \underline{\mathcal{U}}, \quad \operatorname{im}[P^1, P^2] = \overline{\mathcal{R}}, \quad \operatorname{im}[P^1, P^3] = \underline{\mathcal{U}}$$

then $(Q^{-1}AQ, Q^{-1}B, CQ)$ is a Weak Kalman decomposition:

$$\left(\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & C_2 & 0 & C_4 \end{bmatrix} \right)$$

$\rightsquigarrow$ (A_{22}, B_2, C_2) has same input-output behavior as (A, B, C) for $x_0 \in \overline{\mathcal{R}}$

Sequence of ext. reach./restr. unobs. subspaces

Back to switched ODE with jumps and Diracs:
$$\begin{aligned} \dot z &= A_k z + B_k u, \quad z(t_0^-) = x_0 \in \mathcal{X}_0,\\ z(s_k^+) &= J_k^z z(s_k^-) + J_k^v v_k\\ y &= C_k z, \quad y[s_k] = \sum_{i=0}^{\rho_k} C_k^i z(s_k^-)\delta_{s_k}^{(i)} \end{aligned}$$

Lemma (Exact reachable/unobsersable subspaces)

$\mathcal{M}_k^\sigma := \mathcal{R}^\sigma_{[t_0,s_{k+1})}$ and $\mathcal{N}_k^\sigma := \mathcal{U}^\sigma_{(s_k,t_f)}$ are recursively given by:

$$\mathcal{M}_{-1}^\sigma = \mathcal{X}_0, \quad \mathcal{M}_k^\sigma := \mathcal{R}_k + e^{A_k\tau_k}(J_k^x \mathcal{M}_{k-1}^\sigma + \operatorname{im} J_k^v), \quad k = 0,1,\ldots \mathtt{m},$$

$$\mathcal{N}_{\mathtt{m}}^\sigma = \mathcal{U}_m, \quad \mathcal{N}_k^\sigma = \mathcal{U}_k \cap e^{-A_k\tau_k}(((J_k^x)^{-1}\mathcal{N}_{k+1}^\sigma) \cap \mathcal{U}_{k+1}^{\text{imp}}), \quad k = \mathtt{m}-1,\ldots,0,$$

Key fact

For any subspace $\mathcal{V} \subseteq \mathbb{R}^n$ and any $A \in \mathbb{R}^{n\times n}$: $\langle \mathcal{V} \mid A\rangle \subseteq e^{At}\mathcal{V} \subseteq \langle A \mid \mathcal{V}\rangle$

Sequence of ext. reach./restr. unobs. subspaces

Back to switched ODE with jumps and Diracs:

$$\begin{aligned} \dot z &= A_k z + B_k u, \quad z(t_0^-) = x_0 \in \mathcal{X}_0,\\ z(s_k^+) &= J_k^z z(s_k^-) + J_k^v v_k\\ y &= C_k z, \quad y[s_k] = \sum_{i=0}^{\rho_k} C_k^i z(s_k^-)\delta_{s_k}^{(i)} \end{aligned}$$

Definition (extended reach./restricted unobs. subspaces)

$\overline{\mathcal{R}}_k \subseteq \mathcal{R}^\sigma_{[t_0,s_{k+1})}$ and $\underline{\mathcal{U}}_k \subseteq \mathcal{U}^\sigma_{(s_k,t_f)}$ are recursively given by:

$$\overline{\mathcal{R}}_{-1} := \mathcal{X}_0, \quad \overline{\mathcal{R}}_k := \mathcal{R}_k + \langle A_k \mid J_k^x \overline{\mathcal{R}}_{k-1} + \operatorname{im} J_k^v \rangle, \quad k = 0, 1, \ldots \mathtt{m},$$

$$\underline{\mathcal{U}}_{\mathtt{m}} := \mathcal{U}_m, \quad \underline{\mathcal{U}}_k := \mathcal{U}_k \cap \langle ((J_k^x)^{-1} \underline{\mathcal{U}}_{k+1}) \cap \mathcal{U}_{k+1}^{\text{imp}} \mid A_k \rangle, \quad k = \mathtt{m} - 1, \ldots, 0,$$

Key fact

For any subspace $\mathcal{V} \subseteq \mathbb{R}^n$ and any $A \in \mathbb{R}^{n\times n}$: $\langle \mathcal{V} \mid A \rangle \subseteq e^{At}\mathcal{V} \subseteq \langle A \mid \mathcal{V} \rangle$

Reduced realization via weak Kalman decomposition

For each mode k: $\overline{\mathcal{R}}_k$, $\underline{\mathcal{U}}_k$ $\rightsquigarrow$ weak Kalman decomposition:

$$\begin{bmatrix} * \\ W_k \\ * \\ * \end{bmatrix} A_k \begin{bmatrix} * & V_k & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ 0 & \widehat{A}_k & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}, \qquad \begin{bmatrix} * \\ W_k \\ * \\ * \end{bmatrix} B_k = \begin{bmatrix} * \\ \widehat{B}_k \\ 0 \\ 0 \end{bmatrix}$$

$$C_k \begin{bmatrix} * & V_k & * & * \end{bmatrix} = \begin{bmatrix} 0 & \widehat{C}_k & 0 & * \end{bmatrix}$$

$$\widehat{C}_k^i := C_k^i V_{k-1}, \qquad \widehat{J}_k^z := W_k J_k^z V_{k-1}, \qquad \widehat{J}_k^v := W_k J_k^v$$

Reduced sw. ODE with jumps and Diracs:

$$\begin{aligned} \dot{\widehat{z}} &= \widehat{A}_k \widehat{z} + \widehat{B}_k u, \quad \widehat{z}(t_0^-) = \Pi^{\mathcal{X}_0} x_0 \in \widehat{\mathcal{X}}_0, \\ \widehat{z}(s_k^+) &= \widehat{J}_k^z \widehat{z}(s_k^-) + \widehat{J}_k^v v_k \\ y &= \widehat{C}_k z, \quad y[s_k] = \sum_{i=0}^{\rho_k} \widehat{C}_k^i \widehat{z}(s_k^-) \delta_{s_k}^{(i)} \end{aligned}$$

Toy example - reduced realization

on (s_0, s_1) :

$$\dot z = \begin{bmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} z + \begin{bmatrix}0\\0\\1\\0\end{bmatrix} u$$
$$z(s_0^+) = x_0$$
$$y = 0$$
$$y[s_0] = 0$$

on (s_1, s_2) :

$$\dot z = 0$$
$$z(s_1^+) = \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} z(s_1^-)$$
$$y = \begin{bmatrix}0&0&0&1\end{bmatrix} z + \begin{bmatrix}0&-1\end{bmatrix} \begin{pmatrix} u\\ \dot u\end{pmatrix}$$
$$y[s_1] = \left(\begin{bmatrix}0&0&-1&0\end{bmatrix} z(s_1^-) - \begin{bmatrix}1&0\end{bmatrix} \begin{pmatrix} u(s_1^+)\\ \dot u(s_1^+)\end{pmatrix} \right) \delta_{s_1}$$

on (s_2, s_3) :

$$\dot z = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&1&0 \end{bmatrix} z$$
$$z(s_2^+) = \begin{bmatrix} 0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix} z(s_2^-) - \begin{bmatrix}0&0\\0&0\\1&0\\0&1\end{bmatrix} \begin{pmatrix} u(s_1^-)\\ \dot u(s_1^-)\end{pmatrix}$$
$$y = \begin{bmatrix}0&0&0&1\end{bmatrix} z$$
$$y[s_2] = 0$$

$$\overline{\mathcal{R}}_0 = \operatorname{im} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix} \qquad \overline{\mathcal{R}}_1 = \operatorname{im} \begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix} \qquad \overline{\mathcal{R}}_2 = \operatorname{im} \begin{bmatrix}0&0&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\underline{\mathcal{U}}_0 = \operatorname{im} \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad \underline{\mathcal{U}}_1 = \operatorname{im} \begin{bmatrix}1&0\\0&0\\0&1\\0&0\end{bmatrix} \qquad \underline{\mathcal{U}}_2 = \operatorname{im} \begin{bmatrix}1&0\\0&-1\\0&1\\0&0\end{bmatrix}$$

Toy example - reduced realization

on (s_0, s_1) :	on (s_1, s_2) :	on (s_2, s_3) :
$\dot{\widehat{z}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \widehat{z} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ $y = 0$ $y[s_0] = 0$	$\dot{\widehat{z}} = 0$ $\widehat{z}(s_1^+) = \begin{bmatrix} 1 & 0 \end{bmatrix} \widehat{z}(s_1^-)$ $y = \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{pmatrix} u \\ \dot{u} \end{pmatrix}$ $y[s_1] = \left(\begin{bmatrix} 0 & -1 \end{bmatrix} \widehat{z}(s_1^-) - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} u(s_1^+) \\ \dot{u}(s_1^+) \end{pmatrix} \right) \delta_{s_1}$	$\dot{\widehat{z}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \widehat{z}$ $\widehat{z}(s_2^+) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} z(s_2^-) - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} u(s_2^-) \\ \dot{u}(s_2^-) \end{pmatrix}$ $y = \begin{bmatrix} 0 & 1 \end{bmatrix} \widehat{z}$ $y[s_2] = 0$
$\overline{\mathcal{R}}_0 = \operatorname{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$\overline{\mathcal{R}}_1 = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\overline{\mathcal{R}}_2 = \operatorname{im} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\underline{\mathcal{U}}_0 = \operatorname{im} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\underline{\mathcal{U}}_1 = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\underline{\mathcal{U}}_2 = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Impulse decoupling

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned}$$

QWF

$$\begin{aligned} \dot{z} &= A_k^{\text{diff}} z + B_k^{\text{diff}} u, \quad \text{on } (s_k, s_{k+1}) \\ z(s_k^+) &= \Pi_k \left[z(s_k^-) + \boldsymbol{B}_{k-1}^{\text{imp}} \boldsymbol{U}^{\nu_{k-1}}(s_k^-) \right] \\ y &= C_k z + D_k u + \boldsymbol{D}_k^{\text{imp}} \boldsymbol{U}^{\nu_k}, \quad \text{on } (s_k, s_{k+1}) \\ y[s_k] &= \sum_{i=0}^{\nu_k - 2} \left[C_k^i z(s_k^-) + \boldsymbol{D}_{k,i}^{\text{imp}-} \boldsymbol{U}^{\nu_{k-1}}(s_k^-) - \boldsymbol{D}_{k,i}^{\text{imp}+} \boldsymbol{U}^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)} \end{aligned}$$

reduced realization

$$\begin{aligned} \dot{\widehat{z}} &= \widehat{A}_k^{\text{diff}} z + \widehat{B}_k^{\text{diff}} u, \quad \text{on } (s_k, s_{k+1}), \\ \widehat{z}(s_k^+) &= \widehat{\Pi}_k \widehat{z}(s_k^-) + \widehat{J}_k^v \boldsymbol{U}^{\nu_{k-1}}(s_k^-), \\ y &= \widehat{C}_k \widehat{z} + D_k u + D_k^{\text{imp}} U^{\nu_k}, \quad \text{on } (s_k, s_{k+1}), \\ y[s_k] &= \sum_{i=0}^{\nu_k - 2} \left[\widehat{C}_k^i \widehat{z}(s_k^-) + D_{k,i}^{\text{imp}-} U^{\nu_{k-1}}(s_k^-) - D_{k,i}^{\text{imp}+} U^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)}, \end{aligned}$$

impulse decoupling

$$\begin{aligned} \dot{x}^{\overline{\text{imp}}} &= A_k^{\overline{\text{imp}}} x^{\overline{\text{imp}}} + B_k^{\overline{\text{imp}}} u, \quad \text{on } (s_k, s_{k+1}), \\ \dot{x}^{\underline{\text{imp}}} &= A_k^{\underline{\text{imp}}} x^{\underline{\text{imp}}} + B_k^{\underline{\text{imp}}} u, \quad \text{on } (s_k, s_{k+1}), \\ x^{\overline{\text{imp}}}(s_k^+) &= J_k^{x^{\overline{\text{imp}}}} x^{\overline{\text{imp}}}(s_k^-) + J_k^{v^{\overline{\text{imp}}}} \boldsymbol{U}^{\nu_{k-1}}(s_k^-), \\ x^{\underline{\text{imp}}}(s_k^+) &= J_k^{x^{\underline{\text{imp}}}} x^{\underline{\text{imp}}}(s_k^-) + J_k^{v^{\underline{\text{imp}}}} \boldsymbol{U}^{\nu_{k-1}}(s_k^-), \\ y &= C_k^{\overline{\text{imp}}} x^{\overline{\text{imp}}} + C_k^{\underline{\text{imp}}} x^{\underline{\text{imp}}} + D_k u + D_k^{\text{imp}} U^{\nu_k} \\ y[s_k] &= \sum_{i=0}^{\rho_k} \left[C_k^{\text{imp},i} x^{\text{imp}}(s_k^-) + D_{k,i}^{\text{imp}-} U^{\nu_{k-1}}(s_k^-) - D_{k,i}^{\text{imp}+} U^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)}, \end{aligned}$$

midpoint balanced truncation

$$\begin{aligned} \dot{\widehat{x}}^{\overline{\text{imp}}} &= \widehat{A}_k^{\overline{\text{imp}}} \widehat{x}^{\overline{\text{imp}}} + \widehat{B}_k^{\overline{\text{imp}}} u, \quad \text{on } (s_k, s_{k+1}), \\ \dot{x}^{\underline{\text{imp}}} &= A_k^{\underline{\text{imp}}} x^{\underline{\text{imp}}} + B_k^{\underline{\text{imp}}} u, \quad \text{on } (s_k, s_{k+1}), \\ \widehat{x}^{\overline{\text{imp}}}(s_k^+) &= \widehat{J}_k^{\widehat{x}^{\overline{\text{imp}}}} \widehat{x}^{\overline{\text{imp}}}(s_k^-) + \widehat{J}_k^{v^{\overline{\text{imp}}}} \boldsymbol{U}^{\nu_{k-1}}(s_k^-), \\ x^{\underline{\text{imp}}}(s_k^+) &= J_k^{x^{\underline{\text{imp}}}} x^{\underline{\text{imp}}}(s_k^-) + J_k^{v^{\underline{\text{imp}}}} U^{\nu_{k-1}}(s_k^-), \\ \widehat{y} &= \widehat{C}_k^{\overline{\text{imp}}} \widehat{x}^{\overline{\text{imp}}} + C_k^{\underline{\text{imp}}} x^{\underline{\text{imp}}} + D_k u + D_k^{\text{imp}} U^{\nu_k} \\ \widehat{y}[s_k] &= \sum_{i=0}^{\rho_k} \left[C_k^{\text{imp},i} x^{\text{imp}}(s_k^-) + D_{k,i}^{\text{imp}-} U^{\nu_{k-1}}(s_k^-) - D_{k,i}^{\text{imp}+} U^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)} \end{aligned}$$

Overall reduction

Approximation of Dirac impulses?

Assume output Dirac is given by $y[s_k] = C_k^0 z(s_k^-)\delta_{s_k}$

$\rightsquigarrow$ model reduction $\widehat{y}[s_k] = \widehat{C}_k^0 \widehat{z}(s_k^-)\delta_{s_k}$

$\rightsquigarrow$ error $\varepsilon := C_k^0 z(s_k^-) - \widehat{C}_k^0 \widehat{z}(s_k^-)$ leads to output error $y[s_0] - \widehat{y}[s_0] = \varepsilon\delta_{s_0}$

$\rightsquigarrow$ arbitrarily small approximation error leads to infinite error peak

Conclusion for model reduction

Unclear how to quantify error in Dirac impulses (especially for higher order Diracs)

$\rightsquigarrow$ do not reduce parts of states which effect output Diracs

$\rightsquigarrow$ apply further model reduction only on the impulse-unobservable part of the state

Impulse decoupling assumption

For each mode there exists a state decomposition $\mathbb{R}^{n_k} = \mathcal{X}_k^{\mathsf{imp}} \oplus \mathcal{X}_k^{\overline{\mathsf{imp}}}$ s.t.:

1. $\mathcal{X}_{k-1}^{\overline{\mathsf{imp}}} \subseteq \ker[C_k^0 / C_k^1 / \ldots / C_k^{\nu_k - 2}]$
2. $\mathcal{X}_k^{\mathsf{imp}}$ and $\mathcal{X}_k^{\overline{\mathsf{imp}}}$ are A_k-invariant
3. $J_k^z \mathcal{X}_{k-1}^{\mathsf{imp}} \subseteq \mathcal{X}_k^{\mathsf{imp}}$ and $J_k^z \mathcal{X}_{k-1}^{\overline{\mathsf{imp}}} \subseteq \mathcal{X}_k^{\overline{\mathsf{imp}}}$

Midpoint balanced truncation

$$E_\sigma \dot x = A_\sigma x + B_\sigma u,\quad y = C_\sigma x + D_\sigma u$$

QWF

$$\begin{aligned}\dot z &= A_k^{\text{diff}} z + B_k^{\text{diff}} u, \quad \text{on } (s_k, s_{k+1})\\ z(s_k^+) &= \Pi_k\left[z(s_k^-) + \boldsymbol{B}_{k-1}^{\text{imp}} \boldsymbol{U}^{\nu_k-1}(s_k^-)\right]\\ y &= C_k z + D_k u + \boldsymbol{D}_k^{\text{imp}} \boldsymbol{U}^{\nu_k}, \quad \text{on } (s_k, s_{k+1})\\ y[s_k] &= \sum_{i=0}^{\nu_k-2}\left[C_k^i z(s_k^-) + \boldsymbol{D}_{k,i}^{\text{imp}-} \boldsymbol{U}^{\nu_k-1}(s_k^-) - \boldsymbol{D}_{k,i}^{\text{imp}+} \boldsymbol{U}^{\nu_k}(s_k^+)\right]\delta_{s_k}^{(i)}\end{aligned}$$

reduced realization

$$\begin{aligned}\dot{\widehat z} &= \widehat A_k^{\text{diff}} z + \widehat B_k^{\text{diff}} u, \quad \text{on } (s_k, s_{k+1}),\\ \widehat z(s_k^+) &= \widehat\Pi_k \widehat z(s_k^-) + \widehat J_k^v \boldsymbol{U}^{\nu_k-1}(s_k^-),\\ y &= \widehat C_k \widehat z + D_k u + D_k^{\text{imp}} U^{\nu_k}, \quad \text{on } (s_k, s_{k+1}),\\ y[s_k] &= \sum_{i=0}^{\nu_k-2}\left[\widehat C_k^i \widehat z(s_k^-) + D_{k,i}^{\text{imp}-} U^{\nu_k-1}(s_k^-) - D_{k,i}^{\text{imp}+} U^{\nu_k}(s_k^+)\right]\delta_{s_k}^{(i)},\end{aligned}$$

impulse decoupling

$$\begin{aligned}\dot{\overline{x}}^{\text{imp}} &= A_k^{\overline{\text{imp}}} \overline{x}^{\text{imp}} + B_k^{\overline{\text{imp}}} u, \quad \text{on } (s_k, s_{k+1}),\\ \dot{\underline{x}}^{\text{imp}} &= A_k^{\underline{\text{imp}}} \underline{x}^{\text{imp}} + B_k^{\underline{\text{imp}}} u, \quad \text{on } (s_k, s_{k+1}),\\ \overline{x}^{\text{imp}}(s_k^+) &= J_k^{\overline{x}^{\text{imp}}} \overline{x}^{\text{imp}}(s_k^-) + J_k^{\overline{v}^{\text{imp}}} \boldsymbol{U}^{\nu_k-1}(s_k^-),\\ \underline{x}^{\text{imp}}(s_k^+) &= J_k^{\underline{x}^{\text{imp}}} \underline{x}^{\text{imp}}(s_k^-) + J_k^{\underline{v}^{\text{imp}}} \boldsymbol{U}^{\nu_k-1}(s_k^-),\\ y &= C_k^{\overline{\text{imp}}} \overline{x}^{\text{imp}} + C_k^{\underline{\text{imp}}} \underline{x}^{\text{imp}} + D_k u + D_k^{\text{imp}} U^{\nu_k}\\ y[s_k] &= \sum_{i=0}^{\rho_k}\left[C_k^{\text{imp},i} x^{\text{imp}}(s_k^-) + D_{k,i}^{\text{imp}-} U^{\nu_k-1}(s_k^-) - D_{k,i}^{\text{imp}+} U^{\nu_k}(s_k^+)\right]\delta_{s_k}^{(i)},\end{aligned}$$

midpoint balanced truncation

$$\begin{aligned}\dot{\widehat{\overline{x}}}^{\text{imp}} &= \widehat A_k^{\overline{\text{imp}}} \widehat{\overline{x}}^{\text{imp}} + \widehat B_k^{\overline{\text{imp}}} u, \quad \text{on } (s_k, s_{k+1}),\\ \dot{\underline{x}}^{\text{imp}} &= A_k^{\underline{\text{imp}}} \underline{x}^{\text{imp}} + B_k^{\underline{\text{imp}}} u, \quad \text{on } (s_k, s_{k+1}),\\ \widehat{\overline{x}}^{\text{imp}}(s_k^+) &= \widehat J_k^{\overline{x}^{\text{imp}}} \widehat{\overline{x}}^{\text{imp}}(s_k^-) + \widehat J_k^{\overline{v}^{\text{imp}}} \boldsymbol{U}^{\nu_k-1}(s_k^-),\\ \underline{x}^{\text{imp}}(s_k^+) &= J_k^{\underline{x}^{\text{imp}}} \underline{x}^{\text{imp}}(s_k^-) + J_k^{\underline{v}^{\text{imp}}} U^{\nu_k-1}(s_k^-),\\ \widehat y &= \widehat C_k^{\overline{\text{imp}}} \widehat{\overline{x}}^{\text{imp}} + C_k^{\underline{\text{imp}}} \underline{x}^{\text{imp}} + D_k u + D_k^{\text{imp}} U^{\nu_k}\\ \widehat y[s_k] &= \sum_{i=0}^{\rho_k}\left[C_k^{\text{imp},i} x^{\text{imp}}(s_k^-) + D_{k,i}^{\text{imp}-} U^{\nu_k-1}(s_k^-) - D_{k,i}^{\text{imp}+} U^{\nu_k}(s_k^+)\right]\delta_{s_k}^{(i)}\end{aligned}$$

Overall reduction

Notation reset

$$\begin{aligned} \dot{x} &= A_k x + B_k u, && \text{on } (s_k, s_{k+1}), && x(t_0^-) = x_0 \in \mathcal{X}_0, \\ x(s_k^+) &= J_k^x x(s_k^-) + J_k^v v_k, && k \geq 0, \\ y &= C_k x, && \text{on } (s_k, s_{k+1}), \end{aligned}$$

$\Big\downarrow$ reduction

$$\begin{aligned} \dot{\widehat{x}} &= \widehat{A}_k \widehat{x} + \widehat{B}_k u, && \text{on } (s_k, s_{k+1}), && \widehat{x}(t_0^-) = \widehat{x}_0(x_0), \\ \widehat{x}(s_k^+) &= \widehat{J}_k^x \widehat{x}(s_k^-) + \widehat{J}_k^v v_k, && k \geq 0, \\ y &= \widehat{C}_k \widehat{x}, && \text{on } (s_k, s_{k+1}), \end{aligned}$$

Challenge: Two types of inputs

$$\begin{aligned} \dot{x} &= A_k x + B_k u, && \text{on } (s_k, s_{k+1}), && x(t_0^-) = x_0 \in \mathcal{X}_0, \\ x(s_k^+) &= J_k^x x(s_k^-) + J_k^v v_k, && k \geq 0, && \\ y &= C_k x, && \text{on } (s_k, s_{k+1}), && \end{aligned} \tag{swODE}$$

Two types of input

- Continuous input u: Effects $\dot{x} = A_k x + B_k u$ on (s_k, s_{k+1})
- Discrete input v_k: Effects $x(s_k^+) = J_k^x x(s_k^-) + J_k^v v_k$ at switching times s_k

Lemma (Input decoupling)

x *solves* (swODE) $:\Longleftrightarrow$ $x = x_u + x_v$ *where*

- x_u *solves* (swODE) *with* $v_k = 0$ *and* $x_u(t_0^-) = 0$
- x_v *solves* (swODE) *with* $u = 0$ *and* $x_v(t_0^-) = x_0$

Continuous-time Gramians

Definition (Local time-dependent Gramians)

Local reachability Gramian: $P_k(t) := \int_{s_k}^{t} e^{A_k(\tau - s_k)} B_k B_k^\top e^{A_k^\top(\tau - s_k)} \mathrm{d}\tau$

Local observability Gramian: $Q_k(t) := \int_{t}^{s_{k+1}} e^{A_k^\top(s_{k+1}-\tau)} C_k^\top C_k e^{A_k(s_{k+1}-\tau)}$

Definition (Global time-varying Gramians)

- Global reachability Gramian:

$$\boldsymbol{P}_0^\sigma(t) := P_0(t) \text{ for } t \in (t_0, s_1)$$

$$\boldsymbol{P}_k^\sigma(t) := e^{A_k(t-s_k)} J_k^x \boldsymbol{P}_{k-1}^\sigma(s_k^-)(J_k^x)^\top e^{A_k^\top(t-s_k)} + P_k(t) \text{ for } t \in (s_k, s_{k+1})$$

- Global observability Gramian:

$$\boldsymbol{Q}_{\mathtt{m}}^\sigma(t) := Q_{\mathtt{m}}(t) \text{ for } t \in (s_{\mathtt{m}}, t_f)$$

$$\boldsymbol{Q}_k^\sigma := e^{A_k^\top(s_{k+1}-t)}(J_k^x)^\top \boldsymbol{Q}_{k+1}^\sigma J_k^x e^{A_k^\top(s_{k+1}-t)} + Q_k(t) \text{ for } t \in (s_k, s_{k+1})$$

Energy interpretation Gramians

Theorem (Reachability Gramian and input energy)

Consider (swODE) *with* $v_k = 0$ *and* $x_0 = 0$ *and assume that* $\boldsymbol{P}_k^\sigma(t^-)$ *and* $P_k(t)$ *are* *positive definite* *for all* $t \in (t_0, t_f)$. *Then for all* $x_t \in \mathbb{R}^{n_k}$:

$$\min_{\substack{u \text{ s.t.} \\ 0 \overset{u}{\to} x_t}} \int_{t_0}^{t} u(\tau)^\top u(\tau)\,\mathrm{d}\tau = x_t^\top (\boldsymbol{P}_k^\sigma(t^-))^{-1} x_t$$

Theorem (Observability Gramian)

Consider (swODE) *with zero input. Then for all* $t \in (t_0, t_f)$

$$\int_t^{t_f} y(\tau)^\top y(\tau)\,\mathrm{d}\tau = x(t^+)^\top \boldsymbol{Q}_k^\sigma(t^+) x(t^+)$$

Midpoint Gramians

Definition

- Midpoint reachability Gramian: $\overline{\boldsymbol{P}}_k^\sigma := \boldsymbol{P}_k^\sigma(\frac{s_k+s_{k+1}}{2})$
- Midpoint observability Gramian: $\overline{\boldsymbol{Q}}_k^\sigma := \boldsymbol{Q}_k^\sigma(\frac{s_k+s_{k+1}}{2})$

Intuition/Assumption

States which are difficult to reach and observe at midpoint of interval (s_k, s_{k+1}) (quantified by $\overline{\boldsymbol{P}}_k^\sigma$ and $\overline{\boldsymbol{Q}}_k^\sigma$) are also difficult to reach and observe on the whole (finite) time interval.

Midpoint balanced truncation

Use classical balanced truncation for each mode w.r.t. midpoint Gramians

Problem

Effect of discrete input v_k not yet considered!

Discrete time midpoint dynamics

$$\begin{aligned} \dot{x} &= A_k x, && \text{on } (s_k, s_{k+1}), && x(t_0^-) = x_0 \in \mathcal{X}_0, \\ x(s_k^+) &= J_k^x x(s_k^-) + J_k^v v_k, && k \geq 0, \end{aligned} \tag{swODE}$$

Lemma (Solutions at midpoints)

The sequence $x_k^m := x(\frac{s_k+s_{k+1}}{2})$ *of solution midpoints of* (swODE) *satisfy the linear (rectangular) discrete-time system:*

$$x_{k+1}^m = A_k^m x_k^m + B_k^m v_k$$

where

$$A_k^m := e^{A_k \tau_k/2} J_k^x e^{A_{k-1}\tau_{k-1}/2} \in \mathbb{R}^{n_k \times n_{k-1}} \quad \text{and} \quad B_k^m := e^{A_k \tau_k/2} J_k^v$$

Overall midpoint reachability Gramians

Definition (Discrete-time reachability Gramians)

$$\boldsymbol{P}_{-1}^{m} := \gamma X_0 X_0^\top \quad \text{and} \quad \boldsymbol{P}_{k}^{m} = A_k^m \boldsymbol{P}_{k-1}^{m} A_k^{m\top} + B_k^m B_k^{m\top}$$

where X_0 is an orthogonal basis matrix of $\mathcal{X}_0$.

Definition (Overall midpoint reachability Gramian)

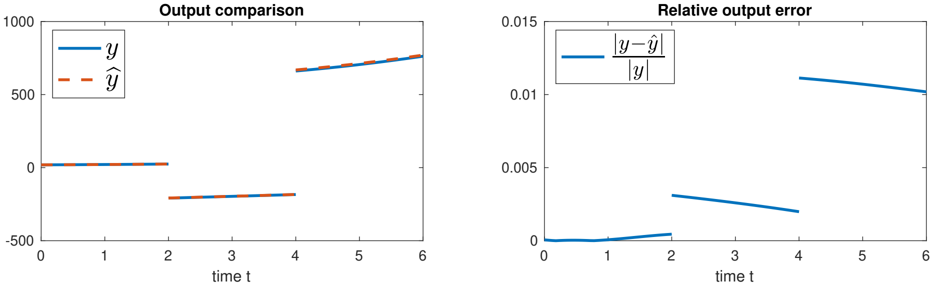
$$\boldsymbol{P}_k^{\lambda} := \overline{\boldsymbol{P}}_k^{\sigma} + \lambda \boldsymbol{P}_k^{m}$$

Role of parameters γ and λ

- γ: How difficult is it to reach the initial value?
- λ: Cost relation between discrete input v_k and continuous input v_k

Medium size academic example

- (swODE) state dimensions: $n_0 = 50$, $n_1 = 60$, $n_2 = 40$
- Coefficient matrices randomly chosen, single input and single output
- Discrete input $v_k = (u(s_k), \dot{u}(s_k))$
- Initial values subspace: $\mathcal{X}_0 = \mathbb{R}^5$
- Reachability Gramian paramters: $\gamma = 0.1$ and $\lambda = 1$
- Hankel singular values threshold: $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0.001$
- Reduced system state dimensions: $\widehat{n}_0 = 8$, $\widehat{n}_1 = 10$, $\widehat{n}_2 = 6$



Summary: Model reduction for switched DAEs

1. **Reduced realization** (always possible, depends only on mode sequence)
   - Via Wong-sequences and Quasi-Weierstrass form rewrite (swDAE) as switched ODE with jumps and impulsive output of same size
   - Calculate extended reachability and restricted unobservability subspaces
   - Calculate weak Kalman decomposition and remove unreachable/unobservable parts
   - Define reduced jump maps, output impulses, initial value space and initial projector
2. **Impulse decoupling** (structural assumption, depends only on mode sequence)
   - Key observation: Dirac impulse = infinite peak
     $\rightsquigarrow$ do not change states which effect output Diracs
   - Assumption: States evolve in two disjoint invariant (mode-dependent) subspaces
3. **Midpoint balanced truncation** (invertability assumption on Gramians)
   - Solution = Solution for continuous input + Solution for discrete input
   - Calculate midpoint reachability Gramians for continuous and discrete time system
   - Calculate midpoint observability Gramians
   - Apply mode-wise balanced truncation via the midpoint Gramians

Remaining challenges and literature

Remaining challenges

- Precise rank decisions required for reduced realization
- Impulse decoupling assumption not constructive
- Large-scale matrix-exponentials are required for midpoint balanced truncation
- Switching signal must be known a-priori

References:

- Hossain & T. (2024): Model reduction for switched differential-algebraic equations with known switching signal, submitted to DAE-Panel
- Hossain & T. (2023): Reduced realization for switched linear systems with known mode sequence, Automatica
- Hossain & T. (2024): Midpoint based balanced truncation for switched linear systems with known switching signal, IEEE TAC